Applications of Chebyshev Polynomials in Numerical Computation

A thesis Submitted in Partial fulfillment of the Requirements for the degree of
M.Sc. in mathematics

Prepared By:
Azkar Abdelrahim Mohieldeen Abdelrahim

Supervisor:
Dr. Mohsin Hassan Abdullah

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قال تعالى:

(وَقَلَّ رَبِّ أَنْخِلْنِي مُدْخَلَ صِدْقٍ وَأَخْرِجْنِي مُخْرَجَ صِدْقٍ وَاجْعَلْ لَيِّ مِنْ لَذُنُكَ سُلْطَانًا نَصِيرًا)

صدق الله العظيم
سورة الإسراء الآية (80)
Dedication

To my Parents and Family
Who love and support me.
And
To my teachers and friends
Who enriched my knowledge.
Acknowledgment

First of all I thank Allah for all the blessings I have day by day. I would like to thank Dr. Mohsin Hassan for his supporting and providing necessary guidance concerning project implementation. Without his superior knowledge and experience, the project wouldn’t like in quality of outcomes, and thus his support has been essential.

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Abstract

This study is concerned with Chebyshev polynomials and their applications in numerical computation. The basic properties of the first kind Chebyshev polynomials in the interval [-1, 1], are used extensively. Three different applications of the first kind Chebyshev polynomials are studied. First, a sufficient condition for convergence of Chebyshev semi-iterative methods applied to the numerical solution of algebraic linear systems is proved. The convergence condition depends on the bounds on the eigenvalues of a square matrix. Second, the problem of approximating a given function by Chebyshev polynomials is considered. The approximating polynomials are used to predict the value of the function at Chebyshev zeros “roots”. Third, the Gauss - Chebyshev quadrature method for the numerical integration of a given function over a finite range is applied. It consists essentially of expanding the integral in a series of Chebyshev polynomials.
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\( T_n \) Chebyshev polynomials of the first kind
\( U_n \) Chebyshev polynomials of the second kind
\( V_n \) Chebyshev polynomials of the third kind
\( W_n \) Chebyshev polynomials of the forth kind
\( T_n^* \) Shifted Chebyshev polynomials of the first kind
\( x_k \) The zeros of Chebyshev polynomials
\( w(x) \) The weight function
\( \langle T_i, T_j \rangle \) The inner product of Chebyshev polynomials
\( f(x) \) Continuous function in the interval \([-1, 1]\)
\( c_i \) The coefficient of Chebyshev polynomials
\( P_n \) Interpolating polynomial of degree \( n \)
\( \rho \) The spectral radius of matrix
\( \lambda_i \) The eigenvalues of matrix
\( J_n \) Interpolating polynomial of degree \( n \)
\( L_k \) Lagrange polynomials
\( A_k \) The coefficients of Gauss – Chebyshev quadrature
\( b_j \) The coefficients of the interpolating polynomial \( J_n \)
\( d_{ij} \) Discrete orthogonality