Laplace Transform Method Solution of Fractional Ordinary Differential Equations

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Abstract

The Laplace transform method has been applied for solving the fractional ordinary differential equations with constant and variable coefficients. The solutions are expressed in terms of Mittage-Leffler functions, and then written in a compact simplified form. As special case, when the order of the derivative is two the result is simplified to that of second order equation.

Keywords: fractional ordinary differential equations, Laplace transform, Mittage-Leffler functions.

Introduction

The intuitive idea of fractional order calculus is as old as integer order calculus. It can be observed from a letter that written by Leibniz to L'Hospital. The fractional order calculus is a generalization of the integer order calculus to a real or complex number. Fractional differential equations used in general in many branches of sciences, mathematics, physics, chemistry and engineering.

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Debnath and Bhatta (Debnath, 2003; Debnath and D Bhatta, 2009),
gave the idea of fractional derivatives and fractional integrals with
their basic properties. Several methods including the Laplace
transform are discussed in introducing the Riemman-Liouville
fractional integral.
They used the fractional derivative to solve the celebrated integral
equation. Ming-Fan and Tao (Ming-Fan and Tao, 2010), introduced
fractional series expansion method to fractional calculus. They
defined a kind of fractional Taylor series of infinitely linear
fractionally differentiable functions, they used this kind of fractional
Taylor series, to give a fractional generalization of hypergeometric
functions, and also they derived the corresponding differential
equations. (Agnieszka and Delfim, 2010) talked about necessary and
sufficient optimality conditions for problems of the linear fractional
calculus of variations with a Lagrangian depending on the free end-
points, where the fractional derivatives are defined in the sense of
Caputo sense.
In this paper we intend to apply Laplace transform method to solve
fractional ordinary differential equations with constant coefficients.
To achieve this task, special formulas of Mittage-Leffler function are
derived and expressed in terms of elementary functions (power,
exponential and error functions), instead of an infinite series. Also
special formulations of inverse Laplace transformation are obtained,
in terms of Mittage-Leffler functions, which already derived. In
obtaining the inverse Laplace transform, some simplified elementary
Algebra, relevant to the derived results is used. This work is based on some basic elements of fractional calculus, with special emphasis on the Riemann-Liouville type. For simplicity, we will mainly use equations of order (2,2) with constant coefficients to illustrate this approach. However equation with variable coefficients of order $\alpha$, where $1 < \alpha \leq 2$, is considered. The obtained solution agrees with the solution of the classical ordinary differential equation, when $\alpha = 2$.

The paper is organized as follows. Section 2, gives a very brief introduction to Lacroix’s formula. Section 3, introduces Liouville’s formulas. Section 4, defines Riemann-Liouville fractional derivatives and integrals, and also the Laplace transformation of fractional integrals is derived. In section 5, the Mittage-Leffler function is defined and the inverse Laplace transform which expressed in terms of Mittage-Leffler functions is described; beside that special types of Mittage-Leffler function which are used further in this paper are obtained. In section 6, fractional ordinary differential equation with constant coefficients is analyzed by using Laplace transform method. As an illustration of the method, two numerical examples of fractional ordinary differential equation with constant coefficients are implemented. Further one example of fractional equation with variable coefficients is solved. Conclusions are given in section 7.
Preliminaries:

Lacroix formula:

In 1819 Lacroix developed the formula for the \( n^{th} \) derivative of \( y = x^m \), where \( m \) is a positive integer. Then

\[
D^n y = \frac{m!}{(m-n)!} x^{m-n}, \quad m \geq n \tag{2.1}
\]

Replacement of the factorial symbol by the gamma function gives:

\[
D^n y = \frac{\Gamma (m+1)}{\Gamma (m-n+1)} x^{m-n} \tag{2.2}
\]

Now (2.2) is defined for either \( n \) integers, or not (arbitrary number) (Debnath and Bhatta. 2009).

Liouville's formula for fractional derivatives:

Liouville's first formula:

For any integer \( n \), we have

\[
D^n e^{ax} = a^n e^{ax} \tag{3.1}
\]

Liouville replaced \( n \) by an arbitrary number \( a \) (rational, irrational or complex), it is clear that the R.H.S of (3.1) is well defined, in this case, he obtained the following formula.

\[
D^a e^{ax} = a^a e^{ax} \tag{3.2}
\]

This formula is called first Liouville's formula. In series expansion of \( f(x) \), Liouville formula is given by
Liouville’s second formula:
Liouville formulated another definition (second form) of a fractional derivative based on the gamma function to extend Lacroix's formula (Lokenath and Bhatta, 2009).

\[ D^\alpha f(x) = \sum_{n=0}^{\infty} c_n \alpha_n \alpha^n x^n \]  

(3.3)

Formula (3.4) is called the Liouville's second definition of fractional derivative. We note that the Liouville derivative of a constant (when \( \beta = 0 \)) is zero, but the derivative of a constant function to Lacroix's formula is

\[ D^\alpha 1 = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \neq 0 \]  

(3.5)

This led to a discrepancy between the two definitions of fractional derivative. But Mathematicians prefer Liouville's definition.

Fractional derivatives and integrals:
The idea of fractional derivatives or fractional integral can be described in different ways. Now by considering a linear homogenous \( n^{th} \)-order ordinary differential equation (initial value problem),

\[ D^n y = 0, \quad y^{(k)}(\alpha) = 0, \quad 0 \leq k \leq n - 1 \]

The solution is the fundamental set \( \{ 1, x, x^2, \ldots, x^{n-1} \} \),
Now we have to derive the Riemann–Liouville formula, that is by seeking the solution of the following inhomogeneous ordinary differential equation [(Lokenath and Bhatta, 2009; Ricardo et al., 2010; Momani and Zaid, 2009):

\[ D^n y = f(x), \quad y^{(k)}(0) = 0, \quad k=0, 1, 2, \ldots \]

To get the solution of the above problem we use the Laplace transform method as follows,

\[ \mathcal{L} D^n \{y\} = \mathcal{L} \{f(x)\} \Rightarrow \bar{y}(s) = s^{-n} \bar{f}(s), \quad \text{where} \]

\[ \bar{y} = \mathcal{L}[y] \text{ and } \bar{f} = \mathcal{L}[f]\]

\[ \therefore y(x) = \mathcal{L}^{-1} s^{-n} \bar{f}(s). \text{By using the convolution theorem, we get the solution as follows:} \]

\[ y(x) = \frac{1}{\Gamma(n)} \int_x^\infty (x-t)^{n-1} f(t) dt, \quad (4.1) \]

Formula (4.1) is called Riemann–Liouville formula. Replacement of \( n \) by a number \( \alpha \) gives the Riemann–Liouville fractional integral.

\[ D^{-\alpha} f(x) \equiv \, aD_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad \text{Re} \alpha > 0 \quad (4.2) \]

Where \( aD_x^{-\alpha} \) (or \( D^{-\alpha} \)) is called the Riemann–Liouville integral operator (Debnath and D Bhatta. 2009 ; Carl and Tom 2000). If
\( \alpha = 0 \) in (4.2), the resulting formula is called Riemann fractional integral and if \( \alpha = -\infty \), is called Liouville fractional integral. The fractional derivative is given by replacing \( \alpha \) by \( -\alpha \) in (4.2).

The Riemann fractional integral is given by,

\[
D^{-\alpha}f(x) \equiv \, \, _{0}D_{x}^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1}f(t)dt, \quad \text{Re} \alpha > 0 \quad (4.3)
\]

The formula (4.3) is of convolution type, then its Laplace transform is given by

\[
\mathcal{L}[D^{-\alpha}f(x)] = \frac{1}{\Gamma(\alpha)} \mathcal{L}[(x^{\alpha-1}) \ast f(x)] = \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{s^{\alpha}} \mathcal{F}(s) = \frac{f(s)}{s^{\alpha}} \quad (4.4)
\]

The Laplace transform of fractional derivative of order \( \alpha \) is given by Lokenath, 2003:

\[
\mathcal{L}[D^{\alpha}x(t)] = s^{\alpha-n}\hat{x}(s) - \sum_{k=0}^{n-1} s^{-k} [D^{(\alpha-k-1)}x(0)] = s^{\alpha-n}\hat{x}(s) - \sum_{k=0}^{n-1} c_{k}s^{-k} \quad (4.5)
\]

Where \( (n - 1) < \alpha \leq n \) and \( c_{k} = D^{(\alpha-k-1)}x(0) \).

**Mittage-Leffler function:**

The special function of Mittage–Leffler function is defined by Haubold *et al.* (Haubold *et al* 2009):

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k + \beta)} \quad (5.1)
\]

Where, \( \beta \in \mathbb{C}, \quad \text{Re}(\alpha), \text{Re}(\beta) > 0. \) If \( \beta = 1 \) then we have,
Some example:

\begin{align*}
(1) \quad E_{0,1}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1)} = \sum_{k=0}^{\infty} z^k = e^z \\
(2) \quad E_{1,1}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = e^z \\
(3) \quad E_{1,2}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \frac{e^z - 1}{z} \\
(4) \quad E_{1,0}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k)} = ze^z
\end{align*}

The function $E(t, \alpha, \alpha)$ is used to solve differential equations of fractional order which is defined by:

\[
E(t, \alpha, \alpha) = t^{\alpha} \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k + \alpha + 1)} = t^{\alpha}E_{1,\alpha+1}(at)
\]  

\textbf{Theorem (1):}

\[
E(t, \alpha, \alpha) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \xi^{\alpha-1} e^{a(t-\xi)} d\xi
\]

\textbf{Proof:} We start by the integral in the left hand side of $(5.4)$,

\[
\frac{1}{\Gamma(\alpha)} \int_{0}^{t} \xi^{\alpha-1} e^{a(t-\xi)} d\xi = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \xi^{\alpha-1} \left( \sum_{k=0}^{\infty} \frac{a^k (t-\xi)^k}{k!} \right) d\xi
\]

\[
= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \left[ \frac{a^k t^k}{k!} \left( \int_{0}^{t} \xi^{\alpha-1} \left( 1 - \frac{\xi}{t} \right)^k d\xi \right) \right]
\]
To complete the proof, we have to calculate the integral given by I, where

\[ I = \int_{0}^{t} \xi^{\alpha-1} \left(1 - \frac{\xi}{t}\right) \, d\xi \]  \hspace{1cm} (2)

Let \( u = \frac{\xi}{t} \), then \( d\xi = t \, du \) as
\[ \xi = 0 \text{ then } u = 0, \text{ and as } \xi = t \text{ then } u = 1. \]

By substituting into (2) we get

\[ I = \int_{0}^{1} t^{\alpha-1} u^{\alpha-1} (1 - u)^k \, t \, du \]

\[ = t^\alpha \int_{0}^{1} u^{\alpha-1} (1 - u)^k \, du = t^\alpha B(\alpha, k + 1) = \frac{t^\alpha \Gamma(\alpha) \Gamma(k + 1)}{\Gamma(\alpha + k + 1)} \]  \hspace{1cm} (3)

Substitution of (3) into (1) yields,

\[ \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \xi^{\alpha-1} e^{a(t - \xi)} \, d\xi = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{a^k t^k}{k!} \frac{t^\alpha \Gamma(\alpha) k!}{\Gamma(\alpha + k + 1)} \]

\[ = t^\alpha \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k + \alpha + 1)} = t^\alpha E_{1,\alpha+1}(at) = E(t, \alpha, \alpha) \]
Corollary (2):

(i) \( E_{\frac{1}{2}}(at) = \frac{e^{at}}{\sqrt{at}} \text{erf}(\sqrt{at}) \)  \hspace{1cm} (5.5)

(ii) \( E_{\frac{1}{2}}(at) = \frac{1}{\sqrt{\pi}} + at e^{at} \text{erf}(\sqrt{at}) \) \hspace{1cm} (5.6)

(iii) \( E_{\frac{1}{2}}(at) = \frac{1}{at} \left[ \text{erf}(\sqrt{at}) - \frac{2}{\sqrt{\pi}} \right] \) \hspace{1cm} (5.7)

(iv) \( E_{\frac{1}{2}}(at) = \frac{-1}{2\sqrt{\pi}} + (at) \left( \frac{1}{\sqrt{\pi}} + \sqrt{at} e^{at} \text{erf}(\sqrt{at}) \right) \) \hspace{1cm} (5.8)

Proof:

(i) Let \( \alpha = \frac{1}{2} \) in (5.4) we obtain,

\[
E\left( t, \frac{1}{2}, a \right) = \frac{1}{\Gamma\left( \frac{1}{2} \right)} \int_0^t \xi^{-\frac{1}{2}} e^{a(t-\xi)} d\xi = \frac{e^{at}}{\Gamma\left( \frac{1}{2} \right)} \int_0^t \xi^{-\frac{1}{2}} e^{-\alpha \xi} d\xi \hspace{1cm} (*)
\]

Let \( u^2 = a\xi \), then \( d\xi = \frac{2udu}{a} \), as \( \xi = 0 \), then \( u = 0 \), and as \( \xi = t \), then \( u = \sqrt{at} \). Substituting into (*) we get,

\[
E\left( t, \frac{1}{2}, a \right) = \frac{e^{at}}{\Gamma\left( \frac{1}{2} \right)} \int_0^\infty \frac{\sqrt{a}}{u} e^{-u^2} 2udu = 2 \frac{e^{at}}{\sqrt{\pi}} \int_0^{\sqrt{at}} e^{-u^2} du = \frac{e^{at}}{\sqrt{a}} \text{erf}(\sqrt{at})
\]

\[
\therefore \sqrt{t} E_{\frac{1}{2}}(at) = \frac{e^{at}}{\sqrt{a}} \text{erf}(\sqrt{at}) \Rightarrow E_{\frac{1}{2}}(at) = \frac{e^{at}}{\sqrt{at}} \text{erf}(\sqrt{at}).
\]
From the definition we have

\[ E_{\frac{1}{2}}(at) = \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma\left(k + \frac{1}{2}\right)} \]  

(5.9)

Replacing \( k \) with \( k + 1 \) in (5.9) we get

\[ E_{\frac{1}{2}}(at) = \sum_{k=-1}^{\infty} \frac{(at)^{k+1}}{\Gamma\left(k + \frac{3}{2}\right)} = \frac{1}{\sqrt{\pi}} + atE_{\frac{3}{2}}(at) = \frac{1}{\sqrt{\pi}} + \sqrt{\pi}te^{at}erf(\sqrt{at}). \]

We used (5.5).

(iii) From the definition we have

\[ E_{\frac{3}{2}}(at) = \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma\left(k + \frac{3}{2}\right)} \]  

(5.10)

Replacing \( k \) with \( k - 1 \) in (5.10) we get

\[ E_{\frac{3}{2}}(at) = \sum_{k=1}^{\infty} \frac{(at)^{k-1}}{\Gamma\left(k + \frac{3}{2}\right)} = (at)^{-1} \left[ \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma\left(k + \frac{3}{2}\right)} - \frac{2}{\sqrt{\pi}} \right] \]

\[ = (at)^{-1} \left[ E_{\frac{3}{2}}(at) - \frac{2}{\sqrt{\pi}} \right] \]

\[ = \frac{1}{at} \left[ e^{at} - \frac{2}{\sqrt{\pi}} \right] \]
We used (5.5).

(iv) From the definition we have

\[ E_{1,-\frac{1}{2}}(at) = \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma\left(k - \frac{1}{2}\right)} \]  

(5.11)

Replacing \( k \) with \( k + 2 \) in (1) we get

\[ E_{1,-\frac{1}{2}}(at) = \sum_{k=-2}^{\infty} \frac{(at)^{k+2}}{\Gamma\left(k + \frac{3}{2}\right)} = (at)^2 \sum_{k=-2}^{\infty} \frac{(at)^k}{\Gamma\left(k + \frac{3}{2}\right)} \]

\[ = (at)^2 \left[ \frac{(at)^{-2}}{\Gamma\left(-\frac{1}{2}\right)} + \frac{(at)^{-1}}{\Gamma\left(\frac{1}{2}\right)} + \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma\left(k + \frac{3}{2}\right)} \right] \]

\[ = \frac{-1}{2\sqrt{\pi}} + \frac{at}{\sqrt{\pi}} + (at)^2 E_{1,\frac{3}{2}}(at) \]

\[ = \frac{-1}{2\sqrt{\pi}} + (at) \left( \frac{1}{\sqrt{\pi}} + \sqrt{at} e^{at} \text{erf}(\sqrt{at}) \right) \]

We used (5.5).

The Laplace transform of the Mittage-Leffler function is given by the following theorem.

**Theorem (3):**

\[ \mathcal{L}^{-1}\left[ \frac{s^{-\alpha}}{s^{\beta} - a} \right] = t^\alpha E_{\beta,\alpha}(at^\beta), \quad |s^\beta - a| < 1 \]  

(5.12)

**Proof:**

Using the definition of the Laplace transform, we have
\[ L\left[t^{\alpha-1}E_{\beta,\alpha}\left(at^\beta\right)\right] = \int_0^\infty e^{-st\,t^{\alpha-1}}E_{\beta,\alpha}\left(at^\beta\right) dt \]

\[ = \sum_{k=0}^\infty \frac{a^k}{\Gamma(k\beta + \alpha)} \int_0^\infty e^{-st\,t^{k\beta + \alpha-1}} dt \]

\[ = \sum_{k=0}^\infty \frac{a^k}{\Gamma(k\beta + \alpha)} \mathcal{L}\left(t^{k\beta + \alpha-1}\right) \]

\[ = \sum_{k=0}^\infty \frac{a^k}{\Gamma(k\beta + \alpha)} \frac{\Gamma(k\beta + \alpha)}{s^{k\beta + \alpha}} = \frac{1}{s^\alpha} \sum_{k=0}^\infty \left(\frac{a}{s^\beta}\right)^k = \frac{s^{-(\alpha-\beta)}}{s^\beta - \alpha} \]

\[ \therefore L^{-1}\left[\frac{s^{-(\alpha-\beta)}}{s^\beta - \alpha}\right] = t^{\alpha-1}E_{\beta,\alpha}\left(at^\beta\right) \]

If \( \beta = 1 \) Then,

\[ L^{-1}\left[\frac{s^{-(\alpha-1)}}{s - \alpha}\right] = t^{\alpha-1}E_{1,\alpha}\left(at\right) = E(t,\alpha - 1,\alpha) \quad (5.13) \]

Corollary (4):

(i) \[ L^{-1}\left[\frac{1}{s^\alpha(s - \alpha)^2}\right] = tE(t,\alpha,\alpha) - \alpha E(t,\alpha + 1,\alpha) \quad (5.14) \]

(ii) \[ L^{-1}\left[\frac{1}{s^\alpha(s - \alpha)^2}\right] = \frac{1}{2} t^2 E(t,\alpha,\alpha) - \alpha t E(t,\alpha + 1,\alpha) + \frac{\alpha(\alpha + 1)}{2} E(t,\alpha + 2,\alpha) \quad (5.15) \]
Proof:

(i) $\mathcal{L}[tE(t, \alpha, \alpha) - \alpha E(t, \alpha + 1, \alpha)] = -\frac{d}{ds} \mathcal{L}[E(t, \alpha, \alpha)] - \alpha \mathcal{L}[E(t, \alpha + 1, \alpha)]$

$$= -\frac{d}{ds} \left[ \frac{s^{-\alpha}}{s - \alpha} \right] - \alpha \left[ \frac{s^{-(\alpha + 1)}}{s - \alpha} \right]$$

$$= \frac{1}{s^\alpha (s - \alpha)^2}$$

(ii) $\mathcal{L}\left[ \frac{1}{2} t^2 E(t, \alpha, \alpha) - t\alpha E(t, \alpha + 1, \alpha) + \frac{1}{2} \alpha(\alpha + 1)E(t, \alpha + 2, \alpha) \right]$

$$= \frac{1}{2} \frac{d^2}{ds^2} \mathcal{L}[E(t, \alpha, \alpha)] + \alpha \frac{d}{ds} \mathcal{L}[E(t, \alpha + 1, \alpha)] + \frac{\alpha(\alpha + 1)}{2} \mathcal{L}[E(t, \alpha + 2, \alpha)]$$

$$= \frac{1}{2} \frac{d^2}{ds^2} \left( \frac{s^{-\alpha}}{s - \alpha} \right) + \alpha \frac{d}{ds} \left( \frac{s^{-(\alpha + 1)}}{s - \alpha} \right) + \frac{\alpha(\alpha + 1)}{2} \left( \frac{s^{-(\alpha + 2)}}{s - \alpha} \right) = \frac{1}{s^\alpha (s - \alpha)^3}$$

Fractional Ordinary differential equations:

The general form of a fractional linear ordinary differential equation of order \((n,q)\) is given by

$$\left[ D^\alpha + a_{n-1} D^{(n-1)\alpha} + \cdots + a_0 D^0 \right] x(t) = h(t), \quad t \geq 0 \quad (6.1)$$

Where $\alpha = \frac{1}{q}$. If $q = 1$, then $\alpha = 1$ and equation (6.1) is a simple ordinary differential equation of order $n$. Symbolically equation (6.1) can be expressed as

$$f(D^\alpha) x(t) = h(t) \quad (6.2)$$

Where $D^\alpha \equiv \left[ D^n + a_{n-1} D^{(n-1)\alpha} + \cdots + a_0 D^0 \right]$ and $a_0, a_1, \ldots, a_{n-1}$ are functions of the independent variable $t$. 
Analysis of the method

Assume that the coefficients $a_0, \ a_1, \ldots, \ a_{n-1}$ are constants in equation (6.1). The by applying the Laplace transformation with respect to $t$ to both sides of (6.1) we obtain,

$$ \mathcal{L}\left[\left(D^{\alpha n} + a_{n-1}D^{(n-1)\alpha} + \cdots + a_0D^0\right)x(t)\right] = \mathcal{L}\{h(t)\} \Rightarrow $$

$$ \mathcal{L}\{D^{\alpha n}\}x(t) + a_{n-1}\mathcal{L}\{D^{(n-1)\alpha}\}x(t) + \cdots + a_0\mathcal{L}\{D^0\}x(t) = \mathcal{L}\{h(t)\} \Rightarrow $$

$$ [s^{n\alpha} \tilde{x}(s) - \sum_{k=1}^{n} c_k s^{nk}] + a_{n-1} [s^{\alpha(n-1)} \tilde{x}(s)] $$

$$ - \sum_{k=1}^{n-1} c_{k-1} s^{k} + \ldots + a_1 s^{\alpha} \tilde{x}(s) + a_0 \tilde{x}(s) = \tilde{h}(s) \Rightarrow $$

$$ [s^{\alpha n} + a_{n-1} s^{\alpha(n-1)} + \cdots + a_1 s^{\alpha} + a_0] \tilde{x}(s) = \tilde{\phi}(s) \Rightarrow $$

$$ \tilde{x}(s) = \frac{\tilde{\phi}(s)}{s^{\alpha n} + a_{n-1} s^{\alpha(n-1)} + \cdots + a_1 s^{\alpha} + a_0} \quad (6.3) $$

If the equation of order $(n, 2)$, then $\alpha = \frac{1}{2}$, and formula (6.3), becomes

$$ \tilde{x}(s) = \frac{\tilde{\phi}(s)}{s^{\frac{n}{2}} + a_{n-1} s^{\frac{n-1}{2}} + \cdots + a_1 s^\frac{1}{2} + a_0} \quad (6.4) $$

Assume that the R.H.S of (6.4) will be factorized and expressed as,
\( \bar{x}(s) = \sum_{r=1}^{k} \frac{\phi_r(s)}{(\sqrt{s} - \beta_r)^{m_r}} = \sum_{r=1}^{k} \frac{\phi_r(s)(\sqrt{s} + \beta_r)^{m_r}}{(s - \beta_r^2)^{m_r}} \\
= \sum_{r=1}^{k} \sum_{i=0}^{m_r} \frac{\phi_r(s)(\sqrt{s})^i(\beta_r)^{m_r-i}}{(s - \beta_r^2)^{m_r}} = \sum_{r=1}^{k} \sum_{i=0}^{m_r} s^{m_r}(s - \beta_r^2)^{m_r} \tag{6.5} \)

Where \( m_1 + m_2 + \ldots + m_k = n, \ \omega_i = \text{const} \) and \( s^{m_r} = \left( \phi_r(s)(\sqrt{s})^{m_r} \right)^{-1}. \)

**Numerical examples:**

**Example 1:** Consider the homogeneous equation of order (2, 2). From (6.1) we have

\( \left( D^1 - 3D^2 + 2D^0 \right) x(t) = 0 \) \( \tag{6.6} \)

Now by applying Laplace transform to (6.6) we have

\( \mathcal{L}[Dx(t)] - 3 \mathcal{L}\left[D^\frac{1}{2}x(t)\right] + 2 \mathcal{L}[D^0 x(t)] = 0 \)

\( \Rightarrow \)

\( s\ddot{x}(s) - x(0) - 3s \ddot{x}(s) + 2\ddot{x}(s) = 0 \)

\( \Rightarrow \)

\( s\ddot{x}(s) - x(0) - 3s^\frac{3}{2}\ddot{x}(s) + 3s^\frac{1}{2}\ddot{x}(0) + 2\ddot{x}(s) = 0 \)

\( \ddot{x}(s) = \left( \frac{c}{-1 + \sqrt{s}} \right) = c \left( \frac{2 + \sqrt{s}}{s - 4} - \frac{1 + \sqrt{s}}{s - 1} \right) \)

\( \Rightarrow \)

\( \ddot{x}(s) = c \left( \frac{2}{s - 4} + \frac{1}{s^{-\frac{1}{2}}(s - 4)} - \frac{1}{s - 1} - \frac{1}{s^{\frac{3}{2}}(s - 1)} \right) \)
Where \( c = [x(0) - 3D^{-\frac{1}{2}}x(0)] \). Using the results given by (5.5), we obtain the solution as follows

\[
\therefore x(t) = c \left( 2e^{4t} - e^t + E(t, -\frac{1}{2}, 4) - E(t, -\frac{1}{2}, 1) \right)
\]

\[
= c \left( 2e^{4t} - e^t + \frac{1}{\sqrt{t}} \left( E_{1,\frac{1}{2}}(4t) - E_{1,\frac{1}{2}}(t) \right) \right)
\]

\[
= c \left( 2e^{4t} \left( 1 + erf(2\sqrt{t}) \right) - e^t \left( 1 + erf(\sqrt{t}) \right) \right)
\]

\[
= c \left[ 2e^{4t}erf(-2\sqrt{t}) - e^t erf(-\sqrt{t}) \right]
\]

**Example 2:** Consider the inhomogeneous initial value problem of order (2,2),

\[
(\dddot{D}^1 - 2D^2 + D^0) x(t) = e^t, \quad x(0) = 1 \quad (6.7)
\]

The application of Laplace transform gives

\[
s \ddot{x}(s) - x(0) - 2s^2 \ddot{x}(s) + 2D^{-\frac{1}{2}}x(0) + \ddot{x}(s) = \frac{1}{s-1} \Rightarrow
\]

\[
(s - 1)^2 \ddot{x}(s) = \frac{1}{s-1} + \left( 1 - 2D^{-\frac{1}{2}}x(0) \right) \Rightarrow
\]

\[
\ddot{x}(s) = \frac{1}{(\sqrt{s} + 1)(\sqrt{s} - 1)^3} + \frac{(1 + c)}{(\sqrt{s} - 1)^2}
\]

\[
= \frac{(1 + c)}{(\sqrt{s} - 1)^2} + \frac{1}{8} \left( \frac{-1}{(\sqrt{s} + 1)} + \frac{1}{(\sqrt{s} - 1)} - \frac{2}{(\sqrt{s} - 1)^2} + \frac{4}{(\sqrt{s} - 1)^3} \right)
\]
Now by taking the inverse Laplace transform and using the results (5.13), (5.14) and (5.15), we have

\[
\chi(t) = \frac{1}{4} e^t + \left( c + \frac{3}{4} \right) L^{-1} \left( \frac{1}{s-1} \right) + \left( \frac{3}{2} + 2c \right) L^{-1} \left( \frac{1}{s^{\frac{1}{2}}(s-1)^2} \right) \\
+ \left( \frac{3}{4} + c \right) t e^t + \frac{1}{2} L^{-1} \left( \frac{1}{s^{\frac{3}{2}}(s-1)^3} \right) + \frac{3}{2} L^{-1} \left( \frac{1}{s-1} \right) \\
+ \frac{3}{2} L^{-1} \left( \frac{1}{s^{\frac{1}{2}}(s-1)^3} \right) + \frac{1}{2} t^2 e^t
\]

\[
= \frac{1}{4} e^t + \left( c + \frac{3}{4} \right) (e^t + te^t) + \left( \frac{3}{2} + 2c \right) \sqrt{t} \left( E_{\frac{1}{2}}(t) + \frac{1}{2} E_{\frac{3}{2}}(t) \right) \\
+ \left( \frac{3}{4} + c \right) te^t + \left( \frac{1}{2} \right) \sqrt{t} \left( \frac{1}{2} E_{\frac{1}{2}}(t) + \frac{3}{2} E_{\frac{3}{2}}(t) + \frac{3}{8} E_{\frac{5}{2}}(t) \right) \\
+ \left( \frac{3}{2} \right) \left( te^t + \frac{t^2}{2} e^t \right) + \left( \frac{3}{8} \right) \frac{3}{2} \left( 4E_{\frac{1}{2}}(t) + 4E_{\frac{3}{2}}(t) - E_{\frac{5}{2}}(t) \right)
\]

Applying the initial condition \( \chi(0) = 1 \) into the above solution gives
Now by substituting the value of $c$ into the above solution, we get

$$x(t) = \frac{e^t}{4} (3t^2 + 12t + 4) + \frac{\sqrt{t}}{4} (9 + 3t)E_{\frac{1}{2}}(t) + \frac{\sqrt{t}}{16} (15 + 12t)E_{\frac{3}{2}}(t)$$

$$+ \frac{\sqrt{t}}{4} E_{\frac{1}{2}-1}(t) - \frac{3}{16} t^2 E_{\frac{3}{2}}(t)$$

Using the values of $E_{\frac{1}{2}}(t)$, $E_{\frac{3}{2}}(t)$, $E_{\frac{5}{2}}(t)$ and $E_{\frac{7}{2}}(t)$ from corollary (2), we get the final solution as follows:

$$x(t) = \frac{1}{16} \left( 4(3t^2 + 12t + 4)e^t + (15 + 48t + 16t^2 - 3t^3)e^t \operatorname{erf}\left(\sqrt{t}\right) + 2 \sqrt{\frac{t}{\pi}} \left( \frac{3t}{t^2 + 8t + 17} \right) \right).$$

**Fractional ordinary differential equation with variable coefficients:**

**Example 3:** Consider the following fractional ordinary differential equation with variable coefficients

$$tD^\alpha(t) + D^{\alpha-1}x(t) + tx(t) = 0, \quad x(0) = 1, \quad 1 < \alpha \leq 2 \quad (6.8)$$

The application of Laplace transform gives

$$- \frac{d}{ds} \mathcal{L}[tD^\alpha(t)] + \mathcal{L}[D^{\alpha-1}x(t)] + \mathcal{L}[tx(t)] = 0 \quad \Rightarrow$$

$$- \frac{d}{ds} \left[ s^\alpha \bar{x}(s) - \sum_{k=0}^{1} s^k D^{\alpha-k-1}x(0) \right] + \left[ s^{\alpha-1} \bar{x}(s) - \sum_{k=0}^{0} s^k D^{\alpha-k-2}x(0) \right]$$

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As special case take \( \alpha = 2 \), then equation (6.6) becomes

\[
t D^2(t) + D x(t) + tx(t) = 0, \quad x(0) = 0, \quad \chi(0) = \chi(0)
\]

we have

\[
\chi(t) = L^{-1}\left[c / \sqrt{1 + s^2}\right] = c J_0(t)
\]

**Conclusions**

The Laplace transformation method has been successfully applied to find an exact solution of fractional ordinary differential equations, with constant and variable coefficients. Some theorems are
introduced; also special formulas of Mittage-Leffler function are derived with their proofs. The method is applied in a direct way without using any assumptions. The results show that the Laplace transformation method needs small size of computations compared to the Adomain decomposition method (ADM), variational iteration method (VIM) and homotopy perturbation method (HPM). The numerical example for equation of variable coefficients shows that the solution in agreement with classical second order equation for $\alpha = 2$.

It is concluded that the Laplace transformation method is a powerful, efficient and reliable tool for the solution of fractional linear ordinary differential equations.

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**References**


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